Strong Blowing into Supersonic Laminar Flows around Two-Dimensional and Axisymmetric Bodies

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A generalized boundary-layer model of laminar flow around supersonic slender bodies with large surface mass transfer is studied, which includes lateral pressure gradients, transverse curvature and viscous-inviscid interaction effects. An integral-method solution is obtained and applied to the case of uniform massive blowing on supersonic semi-infinite wedges and cones where the dividing streamline is straight. For weak blowing rates, an analytical approximation is derived, which has properties analogous to the two-dimensional turbulent boundary-layer solution of Lees and Chapkis. Their cold wall hypersonic blowing similitude (wherein the streamline geometry and induced pressure ratio $p_e/M^2_\infty p_\infty$ are independent of Mach number and profile parameters) is thereby extended to axisymmetric flows; moreover, we show that this similitude continues to apply even under strong blowing conditions. Numerical results are presented for the blowing effect on the streamline pattern, induced pressure field and blown gas layer thickness, with emphasis on the influence of velocity profile and inviscid flow model choice over a wide range of Mach number-wall temperature conditions. Comparisons are also made with available experimental data; these show that upstream influence of the base is appreciable for massive blowing on finite length bodies.

Nomenclature

a = $[\partial(u/u_e)/\partial\eta]_w$
b = $(\partial g/\partial\eta)_w/(1 - g_w)$
c, C_w = $[\partial(p/p_e)/\partial\eta]_w$, $\rho_w\mu_w/\rho_e\mu_e$, respectively
d_i = $(\Delta_i^*/\Delta)_{a\to 0}$
F_1,F_2,F_3,F_4 = velocity profile functions
g,G = H/H_e, total enthalpy profile function, respectively
h,H = static enthalpy, $h + u^2/2$, respectively
L = body length or characteristic dimension
$m,\dot{m}_w,M,M_{Bw}$ = $(\theta^*/\Delta)_{a\to 0}$, injectant mass flux $\rho_w v_w$, Mach number, injection Mach number, respectively
p,P = static pressure, $(p - p_e)/p_e$, respectively
r,Re = local cylindrical radius $r_w + jy\cos\theta_B$, $\rho_e u_e L/\mu_e$, respectively
s_0,S = y_0/x, Δ/x
T = absolute temperature
u,v = velocity components along x, y, respectively
$x,y,\tilde{y},Y$ = distances along and normal to body surface, $y_{j=0}$, transformed normal coordinate, respectively
α = $\epsilon j \cot\theta_B h_w M_{Bw}^2/h_e(1 - M_{Bw}^2)$
β,β_0 = $(\tilde{\delta}/\Delta)_{a\to 0}$, $(\tilde{y}_0/\Delta)_{a\to 0}$, respectively
γ = specific heat ratio
$\delta,\tilde{\delta},\Delta,\Delta_i^*$ = thickness of combined blowing and mixing layer, $\delta_{j=0}$, transformed blowing and mixing layer thickness, "incompressible" displacement thickness $= \int_0^\Delta \left(1 - \frac{u}{u_e}\right) dY$, respectively
η = Y/Δ
λ_e,λ_∞ = $\dot{m}_w/\rho_e u_e$, $\dot{m}_w/\rho_\infty u_\infty$, respectively
μ = coefficient of viscosity
ρ = density
σ = injectant momentum ratio $(\rho_w v_w^2/\rho_\infty U_\infty^2)$
τ,τ_0 = $(1 + 2\cot\theta_B\beta S)^{1/2} - 1$, $\tau(\beta_0)$, respectively
$\theta_B,\theta_{eff},\Delta\theta_0$ = wedge or cone semiangle, effective body angle $\tan^{-1}s_0$, respectively
$\theta^*,\theta_g^*,\theta_E^*,\theta_{P_1},\theta_{P_2},\theta_{P_3},\theta_V^*,\theta_W^*$ =

$$\int_0^\Delta \left(\frac{u}{u_e} - \frac{u^2}{u_e^2}\right) dY, \quad \theta_g^* = \int_0^\Delta \left(g - \frac{u}{u_e}\right) dY,$$

$$\theta_E^* = \int_0^\Delta \frac{u}{u_e}(1 - g)\,dY,$$

$$\theta_{P_1} = \int_0^\Delta \frac{\rho_e}{\rho} P\,dY, \quad \theta_{P_2} = \int_0^\Delta \frac{\rho_e}{\rho}\frac{r_w}{r}\left(\frac{\partial r}{\partial x}\right)_y P\,dY,$$

$$\theta_{P_3} = \int_0^\Delta \frac{\rho_e}{\rho}\frac{r_w}{r} P\,dY,$$

$$\theta_V^* = \int_0^\Delta \frac{u}{u_e}\left(\frac{v - v_w}{u_e}\right) dY, \quad \theta_w^* = \int_0^\Delta \frac{r}{r_w}\frac{v}{u_e}\left(\frac{\partial P}{\partial Y}\right) dY,$$

respectively

Subscripts

e = local inviscid flow conditions
0 = dividing streamline
w = body surface ("wall")
∞ = freestream

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I. Introduction

SURFACE mass transfer can have significant effects on the flowfield and pressure distribution around hypervelocity vehicles. These effects are especially pronounced when ablation occurs under large aerodynamic or radiative heating or when gaseous injection is employed, yielding mass transfer rates ten to a thousand times larger than those associated with ordinary transpiration-cooled boundary-layer flows. While a large amount of data has accumulated for the transpiration-cooling regime, the only experimental results available for massive blowing are those of Hartunian and Spencer[1] and of Bott[2] for laminar flow over wedges and cones at $M_\infty = 4.5$, Fernandez and Zukoski[3] for turbulent flow along a wind-tunnel floor at $M_\infty = 2.6$, and of Gollnick[4] for turbulent flow on a flat plate at $M_\infty \simeq 4$. However, a small body of related theoretical investigations is beginning to accumulate. Analytical solutions for massive blowing into highly idealized shear flows[5,6] have illuminated some general aspects of the problem.

Compressible laminar boundary-layer solutions in the strong blowing limit have been studied by a variety of analytical and numerical methods for both self-similar and nonsimilar flows.[7-11] A comparable study of the turbulent boundary layer for the case of uniform blowing on two-dimensional supersonic wedges has been reported recently by Lees and Chapkis.[12] Other investigators have treated the problem using a purely inviscid flow model approach wherein the viscous effects are idealized by a thin contact discontinuity along the dividing streamline (DSL). Solutions of this type assuming irrotational flow have been obtained by Ting[13] for a finite flat plate and by Aroesty and Davis[14] and Emanuel[15] for a uniformly blown supersonic cone with a straight dividing streamline. Rotational blown gas motion has been investigated for semi-infinite bodies with a thin inviscid boundary layer model using both inverse[16] and direct[17] methods, whereas approximate solutions without thin layer assumptions have been obtained by integral methods for finite wedges and cones.[18,19] With the exception of the anomalous uniform irrotational blowing solution for the cone,[14,15] these inviscid solutions for normal injection all require a nonvanishing favorable pressure gradient and exhibit extreme sensitivity to conditions at the leading edge. Consequently, for uniform blowing on a wedge or cone in supersonic flow with an attached shock, they predict a curved DSL, whereas the experimental evidence[1-4] indicates that it is straight.

There remains considerable uncertainty concerning the relative importance of the viscous and inviscid aspects of the problem and under what conditions each is dominant. The nonvanishing pressure gradient required by a purely inviscid solution suggests that it apply only when the blowing is strong enough to cause either significant upstream influence of the base of finite length bodies or a detached shock envelope with a blunted, highly curved DSL shape. On the other hand, on supersonic slender bodies, viscous mixing effects are significant up to much higher blowing rates than previously suspected, especially on cold walls,[11] since as Lees and Chapkis[12] point out, blowing greatly enhances the viscous-inviscid interaction effects associated with high-speed compressible mixing layers. These considerations suggest that the over-all massive blowing problem for high-speed slender bodies can be tentatively classified into three basic mass transfer regimes, depending on the relative values of skin friction, mass addition and injectant momentum. The first is the usual transpiration-cooling regime $\lambda_\infty \sim 0(Re^{-1/2}) \lesssim 5 \times 10^{-3}$ wherein the skin-friction effect is comparable to that of mass transfer. The second pertains to moderately strong blowing $10^{-2} \lesssim \lambda_\infty \lesssim 1$ wherein the skin friction is negligible compared to injectant mass entrainment, but the momentum remains relatively small. Here, a thin boundary-layer approach is still valid provided that viscous-inviscid interaction and possible axial pressure gradients caused by upstream influence of the base on finite length bodies via the subsonic blowing layer are both taken into account. The third regime entails very large blowing rates $\lambda_\infty \gtrsim 1$ whose momenta are comparable to or greater than $\rho_\infty U_\infty^2$, wherein a detached shock envelope surrounds a blunted, highly curved DSL and thick blown gas layer whose motion is a rotational inviscid flow dominated by large axial and lateral pressure gradients.

Clearly it is very desirable to have a unified theoretical model of massive blowing which includes each of the aforementioned regimes. The present paper is addressed to this problem for the case of high Reynolds number, supersonic laminar flows around two-dimensional or axisymmetric slender bodies with continuously distributed, homogeneous, normal massive injection. A generalized boundary-layer flow model is proposed (Sec. 2), which includes the combined effects of viscosity, compressibility, transverse curvature (TVC), blowing-induced lateral pressure gradients and viscous-inviscid interaction for arbitrary blowing rates. Our main purpose is to explore the consequences of this model under a set of simplifying assumptions and by an approximate integral method solution whose advantages and limitations are fairly well understood. In Sec. 3, application is made of the special case of uniform massive blowing with a straight DSL on semi-infinite wedges and cones, where upstream influence effects are eliminated to permit examination of other features of the problem. Some important similitude properties of this eigensolution in the hypersonic limit are discussed. Numerical results are presented in Sec. 4 for the blowing effect on the flow geometry and pressure field with emphasis on the influence of velocity profile and inviscid flow model choices over a wide range of flight Mach number-wall temperature conditions. Comparisons are also made where possible with the available experimental data, including consideration of the upstream influence effects.

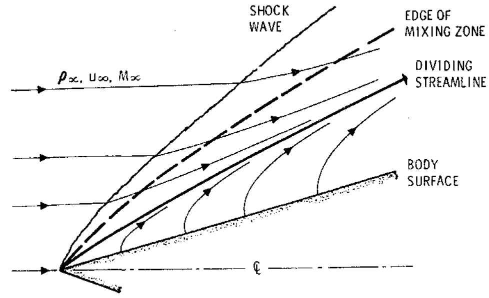

Fig. 1 Flow configuration (schematic).

II. Formulation of Analysis

Physical Description

A schematic illustration of the flowfield on a massively blown slender body in supersonic flow is shown in Fig. 1. Along the body, gas moves outward, and, under the combined action of viscous shear and axial pressure gradients, turns and mixes with the freestream gas which crosses the shock. Within the mixing layer is the dividing streamline (DSL) separating the blown and freestream gases, which presents a thicker effective body shape to the freestream. The location of the DSL, the pressure increase it causes by interaction with the local inviscid flow, and the flow properties across the blown gas layer it overlays are of prime interest in the present study. Provided $\dot{m}_w(x)$ is finite as $x \to 0$, it is noted that the skin-friction effect always dominates near the leading edge regardless of the blowing rate, especially on slender axisymmetric bodies where the skin friction is greatly enhanced by the three-dimensional relief. However, in the present work we shall bypass a detailed investigation of this complicated viscous-inviscid interaction region and focus attention on the flow further downstream where the skin friction and heat transfer have become vanishingly small.

Generalized Boundary-Layer Model

We make the standard simplifying assumptions of a thermally and calorically perfect homogeneous nonreacting mixture with a Prandtl number of unity and a constant density-viscosity ($\rho\mu$) product. The surface temperature is assumed fixed and uniform along the body with a subsonic blowing velocity normal to the surface. Furthermore, longitudinal streamline curvature effects are neglected, thereby restricting the analysis to slender body flows.

In formulating the desired unified theoretical model for arbitrary blowing into high Reynolds number laminar flow, we require that it reduce to the classical boundary-layer equations at small injection rates while possessing the ability to account for lateral pressure gradient effects and to assume a predominantly inviscid flow character at sufficiently strong blowing rates. Guided by an order of magnitude analysis of the Navier-Stokes equations and some results of an idealized shear flow analysis,[6] these considerations naturally lead to the modified boundary-layer equations

$$\partial(\rho u r^\epsilon)/\partial x + \partial(\rho v r^\epsilon)/\partial y = 0 \quad (1)$$

$$\rho u \partial u/\partial x + \rho v \partial u/\partial y + \partial p/\partial x = r^{-\epsilon}(\partial/\partial y)(r^\epsilon \mu \partial u/\partial y) \quad (2)$$

$$\rho u \partial v/\partial x + \rho v \partial v/\partial y + \partial p/\partial y \simeq 0 \quad (3)$$

$$\rho u \partial H/\partial x + \rho v \partial H/\partial y \simeq v \partial p/\partial y + r^{-\epsilon}(\partial/\partial y)(r_\epsilon \mu \partial H/\partial y) \quad (4)$$

where $\epsilon = 0{,}1$ for two-dimensional or axisymmetric flow, respectively, and where the term $v\partial p/\partial y$ in the energy Eq. (4) accounts for the flow work done by the blowing-induced lateral pressure gradient. These equations embody a complete description of the inviscid aspects of the flow; only the viscous effects have been simplified by a boundary-layer approximation wherever they might occur. Such a model is certainly correct for weak blowing and should also be a good approximation for strong blowing where the effects of viscosity lie within a thin blownoff mixing layer if the angular displacement of this layer from the body surface is not too large. It is noted that Eqs. (1–4) have been used recently in investigations of viscous-inviscid interactions in separated flows[20] and viscous mixing processes[21]; the present work constitutes their first application to a surface mass transfer problem.

The foregoing equations are solved subject to the wall boundary conditions of no slip ($u = 0$), prescribed normal mass flux $\dot{m}_w(x)$ and temperature, with p_w an unknown to be determined. The conditions at the outer edge $y = \delta$ of the mixing layer consist of matching u,H and p to their corresponding values in the perturbed inviscid shock layer flow at a local streamline angle $\theta_B + \tan^{-1}(v_e/u_e)$, where v_e/u_e is obtained by integration of Eq. (1). As shown below, the prescribed relationships between u_e,H_e,p_e and v_e/u_e can significantly influence the results of a massive blowing analysis. Note that the need to enforce the outer boundary conditions on u_e and H_e is an implicit viscosity effect which is totally absent from a purely inviscid approach.

It is appropriate at this point to examine the wall compatibility conditions to be used in the subsequent integral method analysis. From Eqs. (1) and (3) and the equation of state evaluated at the surface, for example, the normal pressure gradient at the wall is

$$\left(\frac{\partial p}{\partial y}\right)_w = \left(\frac{\gamma M_{Bw}^2}{1-\gamma M_{Bw}^2}\right)\left[\frac{\epsilon \cos\theta_B}{r_w} - \frac{1}{h_w}\left(\frac{\partial H}{\partial y}\right)_w\right]p_w \quad (5)$$

where $M_{Bw} = (\rho_w v_w^2/\gamma p_w)^{1/2}$ is the injection Mach number. In a similar manner, Eqs. (2) and (4) yield the relations

$$\mu_w\left(\frac{\partial^2 u}{\partial y^2}\right)_w = \frac{dp_w}{dx} + \left[\dot{m}_w - \left(\frac{\partial \mu}{\partial y}\right)_w - \frac{\epsilon\mu_w \cos\theta_B}{r_w}\right]\left(\frac{\partial u}{\partial y}\right)_w \quad (6)$$

$$\mu_w\left(\frac{\partial^2 H}{\partial y^2}\right)_w = -\frac{\dot{m}_w}{\rho_w}\left(\frac{\partial p}{\partial y}\right)_w + \left[\dot{m}_w - \left(\frac{\partial \mu}{\partial y}\right)_w - \frac{\epsilon\mu_w \cos\theta_B}{r_w}\right]\left(\frac{\partial H}{\partial y}\right)_w \quad (7)$$

Equations (6) and (7) are recognized as the usual compressible boundary-layer wall compatibility relations, here generalized to include normal pressure gradients and the TVC effect ($\simeq \epsilon \cos\theta_B$). It is seen that the latter effect acts like an effective suction opposing the effect of injection. Eq. (5) is a new y-momentum relation that explicitly introduces the blowing Mach number [note that the frequently used "incompressible blowing" approximation $M_{Bw} \simeq 0$ relation implies $(\partial p/\partial y)_w \simeq 0$].

Integral Method Analysis

An integral method analysis of Eqs. (1–4) is now carried out along the usual lines. The relative simplicity of this method is especially attractive in cases such as uniform blowing where a self-similar solution of the boundary-layer equations does not exist. Of course, there are some inherent limitations resulting from the average flow description used, namely a loss of detail in the mixing layer and an element of arbitrariness in prescribing the flow property profiles. A more detailed analytical investigation of the model equations (using, e.g., matched inner-outer asymptotic expansion techniques) is required to establish the significance of these limitations.

We introduce a modified Howarth-Dorodnitsyn transformation $dY = \rho r^\epsilon\, dy/\rho_e r_w^\epsilon$ which includes the transverse curvature effect ($r \geq r_w$). For later purposes, we note that inversion of this transformation and integration yields $y = (1 - \epsilon j)\tilde{y} + \epsilon j r_w \sec\theta_B[(r/r_w) - 1]$ where

$$\tilde{y} = y_{j=0} = \int_0^Y \left(\frac{\rho_e}{\rho}\right)dY, \quad \left(\frac{r}{r_w}\right) = \left(1 + 2j\cos\theta_B \frac{\tilde{y}}{r_w}\right)^{1/2}$$

and $j = 0$ or 1 indicates the absence or presence, respectively, of the transverse curvature effect. Integrating Eqs. (1–4) across the combined blown gas and mixing layer region $0 \leq Y \leq \Delta$ and introducing the aforementioned transformation, we obtain the following integral relations for either two-dimensional or axisymmetric flow:

$$(\rho_e r_w^\epsilon)^{-1}\frac{d}{dx}(\rho_e r_w^\epsilon \theta^*) = \lambda_e + \frac{C_w a}{Re\Delta} + \frac{dp_e/dx}{\rho_e u_e^2}\left[\theta^* + \frac{H_e}{h_e}(\theta^* + \theta_g^*)\right] + \frac{p_e}{\rho_e u_e^2 r_w^\epsilon}\left[\frac{d(r_w^\epsilon \theta_{P_1})}{dx} - \epsilon\theta_{P_2}\right] \quad (8)$$

$$\frac{p_e}{\rho_e u_e^2}P_w = (\rho_e r_w)^{-1}\frac{d}{dx}(\rho_e r_w \theta_V^*) + (\Delta - \Delta_i^*)\frac{dv_w/dx}{u_e} + \frac{r_e}{r_w}\left(\frac{v_e - v_w}{u_e}\right)\left(\frac{v_e}{u_e} - \frac{d\delta}{dx}\right) - 2\frac{dp_e/dx}{\rho_e u_e^2}\theta_V^* - \left(\frac{\epsilon j \cos\theta_B}{r_w}\right)\frac{p_e}{\rho_e u_e^2}\theta_{P_3} \quad (9)$$

$$(\rho_e r_w^\epsilon)^{-1}\frac{d}{dx}(\rho_e r_w^\epsilon \theta_E^*) = (1 - g_w)\lambda_e + \frac{C_w b}{Re\Delta} + \frac{dp_e/dx}{\rho_e u_e^2}\theta_E^* - \left(\frac{\gamma - 1}{\gamma}\right)\frac{h_e}{H_e}\theta_W^* + (\theta_E^* + \Delta_i^* - \Delta)\frac{v_e dv_e/dx}{H_e} \quad (10)$$

$$\frac{v_e}{u_e} = \frac{r_w}{r_e}\lambda_e + \frac{d\delta}{dx} - \frac{d/dx[\rho_e r_w^\epsilon u_e(\Delta - \Delta_i^*)]}{\rho_e r_e^\epsilon u_e} \quad (11)$$

where the various integral thicknesses are defined in the nomenclature. Equation (8) is the usual boundary-layer momentum relation with an added axial pressure gradient term due to the normal pressure variation as represented by the "pressure defect" thicknesses θ_{P_1} and θ_{P_2}. The over-all lateral pressure difference is governed by the y-momentum integral Eq. (9), in which appears a new lateral momentum thickness θ_V^*. The energy integral Eq. (10) includes new terms involving the v_e-kinetic energy gradient and the integrated flow work θ_W^* done by the lateral pressure gradient.

Profiles of $u(Y),g(Y)$, and $P(Y)$ are specified by choosing functions that satisfy the boundary and compatibility conditions and provide parametric flexibility without undue complexity. The velocity profile is taken to be $u/u_e = F_1(\eta) + aF_2(\eta)$ with $\eta = Y/\Delta$, where F_1 and F_2 are suitable functions (e.g., polynomials) satisfying the boundary conditions $u(1) = u_e$, $u(0) = 0$, the outer compatibility conditions $\partial u/\partial\eta = \partial^2 u/\partial\eta^2 = \ldots = 0$ at $\eta = 1$, and the compatibility Eq. (6)

which requires that

$$a = \frac{F_1''(0) - (Re\Delta^2 dp_w/dx)/\rho_w u_e^2 C_w}{(Re\Delta/C_w)[\lambda_e - (2\epsilon j\mu_w \cos\theta_B/\rho_e u_e r_w)] - F_2''(0)} \quad (12)$$

Specific choices for F_1 and F_2 depend on the particular problem as discussed below. The pressure defect is assumed to be of the form $P = P_w F_3(\eta) + P'(0)F_4(\eta) + P'(1)F_5(\eta)$ where F_3 and F_4 are chosen so as to satisfy the boundary condition $P(1) = 0$, the outer compatibility condition $P'(1) = 0$ and the compatibility Eq. (5) which yields

$$c = \left(\frac{\partial P}{\partial \eta}\right)_w = \frac{\gamma M_{B_w}^2(1 + P_w)}{1 - \gamma M_{B_w}^2} \times \left[\frac{\epsilon j \cos\theta_B \Delta}{\rho_w r_w/\rho_e} - (1 - g_w)\frac{H_e}{h_w} b\right] \quad (13)$$

Finally, the total enthalpy is expressed in the modified Crocco form $g = g_w + (1 - g_w)u/u_e + (1 - g_w)(b - a)G(\eta)$ where g satisfies the boundary conditions $g(1) = 1$, $g(0) = g_w$, $\partial g/\partial\eta = \partial^2 g/\partial\eta^2 = \ldots = 0$ at $\eta = 1$ and the wall compatibility Eq. (7).

The temperature and density are determined from the relationships $T/T_e = H_e g/h_e - (u_e^2/2h_e)(u/u_e)^2$ and $\rho_e/\rho \simeq T/T_e$, respectively, where the lateral pressure gradient effect on the density profile in the thermal equation of state is neglected (a detailed investigation[22] has shown this to be a good approximation). The normal velocity $v(\eta)$ is calculated by integration and transformation of the continuity equation. Using these profiles, the various integral thicknesses can be evaluated in terms of the basic parameters Δ, P_w, a, b, and c by direct integration. The results, together with Eqs. (12) and (13) and an appropriate inviscid flow model, enable one to calculate the basic unknowns Δ, P_w, b and v_e/u_e from Eqs. (8–11). The DSL location $Y_0(x)$ can then be determined from injectant mass conservation as

$$\rho_e r_w^\epsilon \int_0^{Y_0} u dY = \int_0^x \dot{m}_w r_w^\epsilon dx \quad (14)$$

Equations (8–11) indicate that the usual initial conditions on Δ and b must be supplemented by a second condition on Δ (or $d\Delta/dx$) in the presence of free viscous-inviscid interaction where p_e is unknown a priori. In the solution obtained below, for example, this requirement is met by specifying that the DSL become straight downstream. Furthermore, consideration of the lateral pressure gradient requires a condition on the wall pressure, as discussed below.

III. Asymptotic Solution for Uniform Massive Blowing

General Case

The foregoing analysis is now applied to the important special case of uniform massive blowing on semi-infinite wedges and cones in supersonic flow. Since it has been observed for this case[1,2,4] that the DSL becomes straight downstream of the tip, asymptotic solutions with this property were sought. Such a solution pertains to uniform (but blowing-dependent) supersonic local inviscid flow in the blowing regime where the shock is straight, i.e., where $\theta_B + \tan^{-1}(v_e/u_e)$ is below the detachment angle.

The aforementioned asymptotic solution was obtained by applying the above integral relations to wedges or cones ($r_w = x \sin\theta_B$) under the requirements that y_0/x = const and that the various form factors $a, b, c, \theta^*/\Delta, \Delta_i^*/\Delta$ remain bounded for large $\lambda_e x$. This yields a self-consistent solution with the properties $a \to 0$ and $v_e/u_e, \rho_e, p_e, h_e, u_e$ each constant. Two possible surface pressure behaviors at large x were found, one in which p_w vanishes independently of λ_e and the other in which $p_w - p_e$ approaches a constant determined by the y-momentum integral. On physical grounds, the latter was deemed appropriate for a semi-infinite body.‡ The final solution is a Reynolds number–independent conical flow with negligible skin friction ($a \to 0$, Eq. 22) and pressure gradient governed by the following set of coupled algebraic equations:

$$(1 + \epsilon)mS \simeq \lambda_e + S(p_e/\rho_e u_e^2)(Q_1 P_w + \gamma Q_2 \alpha S) \quad (15)$$

$$(p_e/\rho_e u_e^2)(1 + \epsilon j Q_3 S \cot\theta_B)P_w \simeq [v_e/u_e - (\rho_e/\rho_w)\lambda_e][\lambda_e - (1 + \epsilon) \times (1 - d_i)S] + Q_4 S^2 \quad (16)$$

$$b = (\gamma - 1)g_w \alpha S/(1 - g_w) \quad (17)$$

$$v_e/u_e = (1 - \epsilon j)\beta S + \epsilon j\tau \tan\theta_B + (1 + \epsilon j\tau)^{-1} \times [\lambda_e - (1 + \epsilon)(1 - d_i)S] \quad (18)$$

where $Q_1, \ldots, Q_4$ are nondimensional parameters depending on $\gamma, T_w/T_e$, and M_e^2 (see Ref. 22) and $\alpha = \epsilon j \cot\theta_B h_w M_{B_w}^2/h_e(1 - M_{B_w}^2)$ represents the injection compressibility effect introduced by transverse curvature which introduces a singularity for sonic injection on a cone. Equation (17) shows that the asymptotic heat-transfer parameter b for the cone does not vanish with massive injection when the inviscid work heating associated with blowing of a compressible gas is taken into account. The asymptotic DSL slope in the physical plane is

$$s_0 = (1 - \epsilon j)\beta_0 s + \epsilon j \tau_0 \tan\theta_B \quad (19)$$

where $\beta_0 = \bar{y}_0(\eta_0)/\Delta$ and η_0 is determined from the condition

$$\lambda_e S = (1 + \epsilon)\int_0^{\eta_0} F_1 d\eta.$$

To complete the solution, specific profile functions are introduced which are appropriate to strong blowing and zero pressure gradient. The arbitrariness of profile selection inherent in the integral method can be taken advantage of here to evaluate the sensitivity of the solution to this selection. Accordingly, two alternative forms of the velocity function $F_1(\eta)$ were used, either fourth and sixth degree polynomials á la Kármán-Pohlhausen in which the wall gradients vanish algebraically with strong blowing, or an appropriately normalized Emmons and Leigh[7] blowoff profile wherein all wall derivatives vanish exponentially. The pressure profile functions F_3 and F_4 are represented by third degree polynomials, the simplest functions that can satisfy the conditions cited previously while permitting both the pressure and normal pressure gradient at the wall to float free as unknowns. Finally, we take $G = F_2(\eta)$ as either fourth and sixth degree polynomials or a suitably normalized zero blowing Blasius profile.[7]

Boundary-Layer Limit: Similitude Considerations

Equations (15–18) must be solved numerically for arbitrary values of λ_e (typical results are given below); however, an approximate analytical solution is possible when $\lambda_e \ll 1$. In this limit $S \sim 0(\lambda_e)$ and $P_w \sim \lambda_e^2$ so that the lateral pressure gradient and blowing Mach number effects may be neglected, giving from Eqs. (15) and (18), respectively, that $S \simeq \lambda_e/(1 + \epsilon)m$ and

$$\frac{v_e}{u_e} \simeq \lambda_e\left(\frac{H_e}{h_e} + \frac{h_w d_i}{m h_e}\right) - \frac{\epsilon}{2}\left(\frac{\beta\lambda_e}{m}\right) - \frac{\epsilon j}{2}\left\{\left[\frac{\beta}{m} + \left(\frac{2\tau}{1+\tau}\right)\left(\frac{m + d_i - 1}{m}\right)\right]\lambda_e - 2\tau\tan\theta_B\right\} \quad (20)$$

$$s_0 \cong \lambda_e(\beta_0/m) - \frac{\epsilon}{2}\left(\frac{\beta_0\lambda_e}{m}\right) - \frac{\epsilon j}{2}\left[\left(\frac{\beta_0\lambda_e}{m}\right) - 2\tau_0\tan\theta_B\right] \quad (21)$$

‡ Following Klineberg,[23] the present solution is thus an eigensolution wherein the free interaction-induced branching solutions are suppressed (inviscid approximations to the latter are treated by Thomas[18] and Taylor[19]).

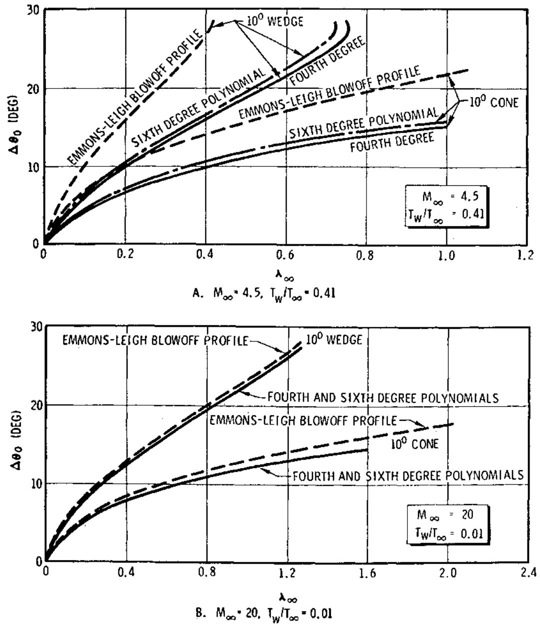

Fig. 2 Influence of velocity profile on dividing streamline angle.

with

$$\beta_0 \cong \int_0^{\eta_0} F_1 d\eta + \frac{h_w}{h_e}\left(1 - \int_0^{\eta_0} F_1 d\eta\right) + \left(\frac{\gamma - 1}{\gamma}\right) M_e^2 \int_0^{\eta_0} (F_1 - F_1^2) d\eta \quad (22)$$

where $\beta \cong 1 - d_i + (h_w d_i/h_e) + m(\gamma - 1)M_e^2/2, H_e/h_e \cong 1 + (\gamma - 1)M_e^2/2$ and where the TVC terms have been retained owing to their large effect for slender cones even when $\lambda_e \ll 1$. Equations (20–22) correspond to the results of Inger[10] for the strong blowing limit of the Kármán-Pohlhausen method solution and for $\epsilon = 0$ are the laminar flow counterparts of the turbulent boundary-layer results of Lees and Chapkis.[12] Indeed, for strongly hypersonic flow past "cold" wedges, Eq. (20) yields exactly their result for the inviscid flow deflection angle in two-dimensional flow.

Equations (20–22) clearly illustrate the profound influence of viscous mixing on the streamline geometry (and by inference, pressure) around slender bodies with massive blowing. It is seen that enhancement of the surface mass transfer effect by viscous dissipation heating and the compressibility associated with $T_w/T_e \neq 0$ are important features at high speeds. The three-dimensional relief also reduces pressures far below those predicted for two-dimensional flows or for axisymmetric flows using only the Mangler transformation ($j = 0$).

Consider now the similitude properties of the foregoing solution for the limiting case of hypersonic small disturbance flow $[H_e/h_e \simeq (\gamma - 1)M_\infty^2/2 \simeq \beta/m \gg 1, u_e \simeq u_\infty, \rho_e/\rho_\infty \simeq p_e/p_\infty]$ past cold ($h_w/H_e \ll 1$) slender bodies. Then with $\lambda_e \simeq p_\infty\lambda_\infty/p_e, p_e/p_\infty \simeq M_\infty^2(v_e/u_e)^2$ and assuming $2\tau(1 - m - d_i) \ll (\tau - 1)m(1 + \tau)M_\infty^2$, Eq. (20) yields $v_e/u_e \simeq (p_e/p_\infty)^{1/2}M_\infty^{-1} \simeq [(1 - \epsilon/2 - \epsilon j/2)(\gamma - 1)M_\infty^2\lambda_\infty p_\infty/2p_e] + \epsilon j\tau \tan\theta_B$ so that

$$\frac{p_e}{p_\infty} \simeq M_\infty^2 \left[\left(\frac{\gamma - 1}{2}\right)\lambda_\infty\right]^{2/3}\left[1 - (1 + j)\frac{\epsilon}{2} + \left(\frac{2\epsilon j}{\gamma - 1}\right)\frac{p_e\,\tau\,\tan\theta_B}{p_\infty\lambda_\infty M_\infty^2}\right]^{2/3} \quad (23)$$

$$\frac{v_e}{u_e} \simeq \left[\left(\frac{\gamma - 1}{2}\right)\lambda_\infty\right]^{1/3}\left[1 - (1 + j)\frac{\epsilon}{2} + \left(\frac{2\epsilon j}{\gamma - 1}\right)\frac{p_e\,\tau\,\tan\theta_B}{p_\infty\lambda_\infty M_\infty^2}\right]^{1/3} \quad (24)$$

where $1 + \tau = [1 + (\gamma - 1)M_\infty^2\lambda_\infty p_\infty \cot\theta_B/2p_e]^{1/2}$. It is seen that these expressions are independent of the boundary-layer profile parameters. Moreover, provided the TVC effect is small enough that $(\gamma - 1)\lambda_\infty^{1/3}\cot\theta_B \ll 2$, they imply the similitude properties $v_e/u_e = \lambda_\infty^{1/3}f_1(\gamma,\epsilon)$, $p_e/p_\infty = M_\infty^2\lambda_\infty^{2/3}f_1^2$, and $\tau = \lambda_\infty^{1/3}\cot\theta_B f_2(\gamma,\epsilon)$ where f_1 and f_2 are independent of M_∞ and the profile parameters. The DSL slope s_0 from Eq. (21) obeys a similitude of the same form as

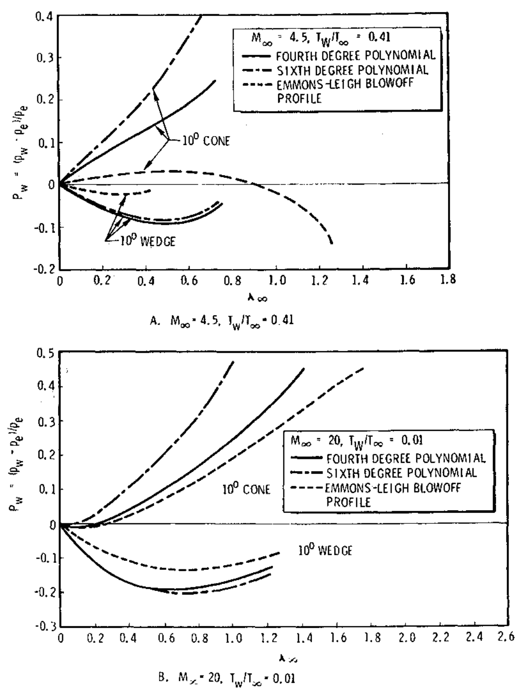

Fig. 3 Over-all pressure difference across boundary layer.

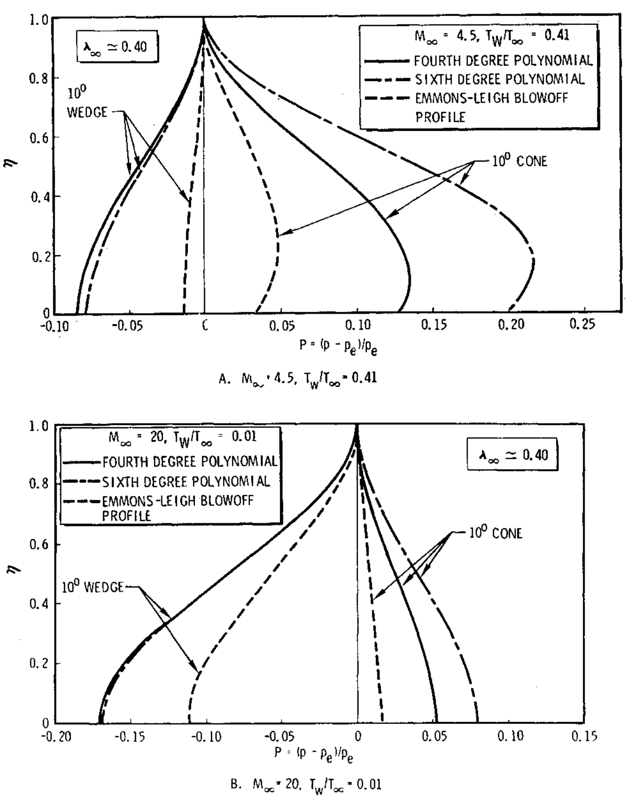

Fig. 4 Typical pressure variations across boundary layer.

v_e/u_e, except that s_0 is no longer strictly independent of the profile parameters.

The foregoing boundary-layer approximations were found to be in close agreement with solutions of the full Eqs. (15–18) over a wide range of wall temperature-Mach number conditions for blowing rates up to $\lambda_\infty \lesssim 1$. Consequently, the lateral pressure gradients, although significant for massive blowing on axisymmetric bodies, generally have a negligible influence on the flow geometry in the attached shock regime.

IV. Numerical Results and Discussion

Profile Sensitivity Study

The influence of velocity profile on the variation of DSL angle $\Delta\theta_0 = \tan^{-1} s_0$ with λ_∞ is shown in Fig. 2 for several Mach number-wall temperature conditions. In general, the two polynomial profiles gives essentially the same results regardless of flight conditions. For the lower Mach number moderately cooled wall case, however, the Emmons-Leigh blowoff profile predicts an appreciably larger DSL angle and smaller detachment λ_∞ than do the polynomials owing to the greater momentum defect of the blowoff profile. For the high Mach number highly cooled wall case, the profile choice has only a small effect, indicating that the hypersonic similitude for $\lambda_\infty \ll 1$ discussed previously can in practice be extended to much stronger blowing rates $\lambda_\infty \lesssim 0\ (1)$ and that the profile-independence aspect of this similitude is approximately true for the DSL angle as well as v_e/u_e. This is further corroborated by our extensive numerical studies[22] showing that the blowing effect on $\Delta\theta_0$ becomes independent of M_∞ in the hypersonic cold wall limit.

The effect of profile choice on the over-all lateral pressure difference is shown in Fig. 3. Again, the results are far more sensitive to the assumed profile at lower Mach number, hot wall flight conditions than at hypersonic cold wall conditions. The most important feature to be observed here is that the over-all pressure difference on a cone is usually opposite in sign to the outward compression predicted for a wedge. This is a result of the lateral streamline spreading, which causes an expansive pressure drop across the cone boundary layer that exceeds the outward compression associated with the turning of the injected gas. The magnitude of the net lateral pressure change depends strongly on the Mach number-surface temperature condition. Some typical pressure distributions across the boundary layer are shown in Fig. 4. Aside from different wall values, the wedge and cone profiles have the same general shape, the latter exhibiting an inflection point near the surface owing to the nonvanishing normal pressure gradient (Eq. 5).

Comparisons with Experiment

A careful experimental study of uniform massive air injection on an 18-in.-long flat plate in a $M_\infty \cong 3.64$ to 3.93 supersonic flow at $Re_\infty/\text{ft} \simeq (6 \text{ to } 12) \times 10^5$ has been recently reported by Gollnick.[4] Unfortunately, a direct comparison with

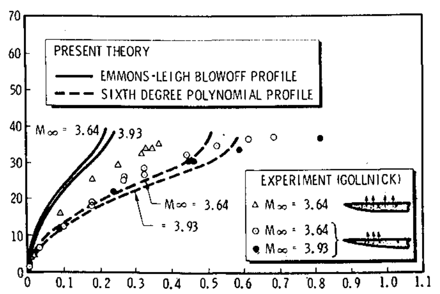

Fig. 5 Comparison of dividing streamline angle predictions with $M_\infty \simeq 4$ flat plate experiments.

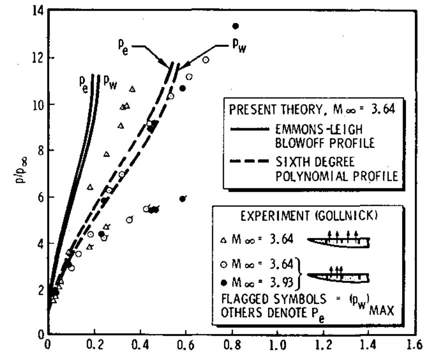

Fig. 6 Comparison of pressure predictions with $M_\infty \simeq 4$ flat plate experiments.

his results is clouded by three features which are not included in the present theory: a) only a portion of the plate near mid-length was injected, leading to appreciable local axial pressure gradients due to upstream influence of the unblown downstream region; b) the mixing layer around the DSL was undoubtedly turbulent for the prevailing test conditions; c) in some runs, schlieren photographs revealed a supersonic injection shock near the surface. Nevertheless, some instructive qualitative comparisons are possible.

The predicted DSL angle variations with blowing rate are compared to Gollnick's results§ for two different injection region lengths (4.5 and 9.0 in., respectively) in Fig. 5. Note that the data indicates a thinner blowing layer on the shorter injection length, which can be explained by the larger upstream influence of the unblown downstream region in this case. The predicted parametric trends with λ_∞ and M_∞ are in qualitative agreement with those of the data. Quantitatively, the semi-infinite body blowoff profile solution falls above the experimental results for the longer injection region, whereas the polynomial profile solution lies nearer the shorter injection region data. This is to be expected under massive blowing conditions, since the larger axial pressure gradients along the shorter injection length would give a velocity profile more nearly of polynomial type than one with exponentially small velocities near the surface.[6]

A comparison of theoretical and experimental pressure data is shown in Fig. 6. It is seen that the induced inviscid pressures are in good qualitative agreement and differ in a manner completely consistent with the foregoing DSL results. On the other hand, although the theory and experiment both indicate a compression ($p_e > p_w$) across the boundary layer, the semi-infinite body wall pressure predictions are much higher than the measured maximum surface values when $\lambda_\infty \gtrsim 0.10$, undoubtedly due to the strong favorable pressure gradients along the injection surface in the experiments. Taken as a whole, this figure also illustrates the large increases in pressure that massive blowing can induce in even a marginally hypersonic flow.

The experiments of Hartunian and Spencer[1] and of Bott[2] dealt with uniform massive blowing on small (3-in. long) wedge and cone models at low Reynolds numbers ($Re_\infty/\text{ft} \sim 150$). Although the large upstream influence effects in their studies certainly obscure any quantitative comparisons with the present theory, a qualitative comparison is still of value. For this purpose, a sixth-degree polynomial profile has been used.

The predicted DSL angle variations with blowing rate for three wedge flows are shown in Fig. 7. For each wedge angle, results are given for the complete tangent wedge model wherein the variations of u_e, T_e and M_e with λ_∞ are taken into account and also for a frequently used hypersonic small disturbance approximation to this model in which these variations are neglected. Both calculations show that the increase

§ For uniform injection Gollnick[4] observed the DSL to be essentially straight throughout the injected region.

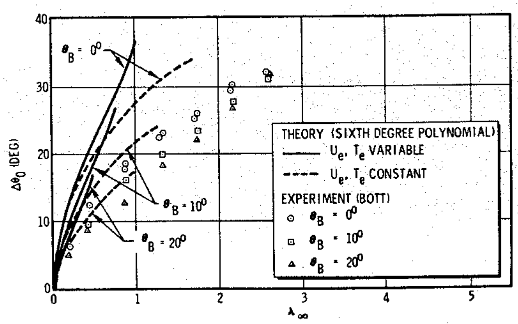

Fig. 7 Comparison of dividing streamline angle predictions with $M_\infty \simeq 4.5$ wedge experiments.

in pressure with wedge angle decreases the blowing layer thickness. However, when λ_∞ is not small compared to unity, there is a marked divergence in the predicted blowing effect; the sharply convex curves of the approximate model fall increasingly below the concave-shaped variations given by the full tangent wedge model and yields much larger shock detachment values of λ_∞. The explanation of these differences lies in the relationship $\lambda_\infty = (\rho_e u_e/\rho_\infty U_\infty)\lambda_e$ and the fact that for increasing λ_e, u_e/U_∞ decreases much more rapidly than does ρ_∞/ρ_e at the larger flow deflection angles; consequently, progressively smaller values of λ_∞ are obtained for a given λ_e than in the case assuming u_e and T_e constant with $\rho_e \sim p_e$.

Also shown in Fig. 7 for comparison are the data of Bott,[2] who carried out an improved version of Hartunian and Spencer's[1] wedge flow experiments. Although the approximate tangent wedge model results are in fair agreement with his data, the complete tangent wedge calculations beyond $\lambda_\infty \approx 0.10$ fall increasingly too high. This illustrates the importance of the inviscid flow model in massive blowing analyses; neglect of the u_e and T_e variations with blowing can significantly underestimate the blown gas layer thickness when $\lambda_\infty \gtrsim 0(1)$ and yield misleading comparisons with experiment. From the results of Fernandez and Zukoski[3] and Gollnick[4] a significant part of the discrepancy between theory and experiment shown can again be attributed to the very short test models used, for which a significant upstream influence of the base exists under massive blowing conditions owing to the thick subsonic gas layer adjacent to the surface. Indeed, large favorable wall pressure gradients up to three quarters of a model length upstream of the base have been observed,[1,2] and these would significantly reduce the blown gas layer thickness (especially at higher blowing rates) relative to the predictions of a semi-infinite body zero pressure gradient solution.

Figure 8 presents a similar comparison of theory and experiment[1] for a 10° cone. Also shown are the comparable wedge and cone solutions without TVC to illustrate the large blown gas layer reduction caused by the TVC effect under massive blowing conditions. Because the three-dimensional relief and mass transfer effects tend to cancel each other for a cone, the DSL angle increases slowly beyond $\lambda_\infty \approx 1$ and lies well below the shock detachment value for all blowing rates. Moreover, the greatly reduced inviscid streamline displacement and pressure on the cone diminishes considerably the difference between the exact and approximate tangent cone model results. The agreement with experiment is somewhat better than for the comparable wedge flow, although beyond $\lambda_\infty \approx 1$, the theory still falls above the data. The much thinner blowing layer on the cone implies less upstream influence and hence somewhat better agreement with a semi-infinite body analysis than obtained in two-dimensional flow, which is consistent with the smaller pressure gradients observed[1] on the cone models.

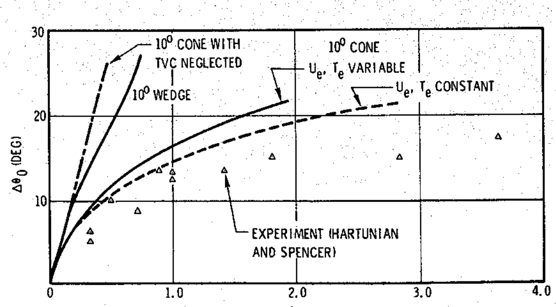

Fig. 8 Comparison of dividing streamline angle predictions with $M_\infty \simeq 4.5$, $\theta_B = 10°$ cone experiments.

V. Summary and Conclusion

Within the framework of an integral method approach, a theoretical study of strong blowing into high Reynolds number laminar flows has been made based on a generalized boundary-layer model which includes the effects of lateral pressure gradient, transverse curvature and viscous-inviscid interaction. An asymptotic "eigensolution" of this model was obtained for the important special case of uniform blowing on supersonic semi-infinite wedges and cones where the dividing streamline is straight and the local skin friction and axial pressure gradient are both negligible. A detailed analytical and numerical study of the solution lead to the following major conclusions.

1) The "weak-blowing" ($\lambda_\infty \ll 1$) hypersonic cold-wall similitude of Lees and Chapkis[12] for two-dimensional flow can be extended to the axially symmetric case and can be applied with good approximation even at fairly high mass transfer rates $\lambda_\infty \lesssim 0(1)$. The dominant influence of viscous dissipation and wall heating on blowing-induced viscous-inviscid interaction observed for turbulent flow[12] is also discerned in the present laminar analysis.

2) The transverse curvature effect has an important influence on the streamline geometry and induced pressure field along slender cones with strong blowing; the use of the Mangler transformation alone would profoundly overestimate these properties. Indeed, because of the counterbalancing between the mass transfer and three-dimensional relief effects, the DSL angle becomes nearly independent of blowing rate for $\lambda_\infty > 1$ and lies well below the shock detachment deflection angle.

3) Massive blowing effects were found to be relatively insensitive to the velocity profile at high Mach number-cold wall flight conditions but very sensitive at lower Mach numbers $M_\infty \lesssim 10$ and higher wall temperatures $T_w/T_\infty \gtrsim 0(1)$. Consequently, the largest differences between laminar and turbulent massive blowing analyses should occur in the DSL locations on "hot" slender cones at lower supersonic Mach numbers. Concerning the influence of the inviscid flow model, it was found that the hypersonic small disturbance approximation yields significant errors at Mach numbers $M_\infty \lesssim 5$; however, this can be appreciably mitigated in practice by upstream influence effects on finite length bodies.

4) The present theory predicts an outward compression of the injected gas on a wedge, as opposed to an *expansion* in the case of a cone. However, although significant in an absolute sense, these lateral pressure differences are usually only a small fraction of the corresponding induced inviscid pressure levels when $\lambda_\infty \lesssim 1$ and do not significantly influence the streamline geometry in the attached shock regime. This influence could be larger for the greater momenta associated with low molecular weight injectants; and a generalization of our analysis to include nonhomogeneous injection effects would be of interest to assess this point.

5) Comparisons of existing theoretical analysis with available massive blowing experiments are clouded by the presence of large favorable axial pressure gradients associated with upstream influence effects of the base on the test models.

Clearly, extension of the present approach to include these effects is desirable. Nevertheless, the general physical features of the present theory as concerns the assumed straightness of the DSL and the parametric trends with λ_∞, M_∞ and flow geometry are in qualitative agreement with the data.

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